Lecture 5: Planet migration

In this final lecture we will look at the early evolution of planets and planetary systems. Once planets attain significant mass they interact gravitationally with the gas disc, and also with any remaining planetesimals and other planets. These processes can lead to substantial changes in the dynamical state of the system, and if we wish to compare to observations of evolved planetary systems we must consider their effects.

1 Migration in gaseous discs

We first consider the interaction between a planet and its parent gas disc. The simplest way to model planet migration is via the so-called impulse approximation. We assume that a small parcel of gas initially moves on an unperturbed circular orbit, and treat the interaction as a two-body problem to calculate the impulsive deflection during a close encounter with the planet. This approach, first used in this context by Lin & Papaloizou (1979), glosses over many details, but it gives the correct scalings and has been shown to provide a reasonably accurate (factor of a few) approximation to more detailed calculations.

If a fluid element in the disc approaches a planet of mass M_{p} with relative velocity Δv and impact parameter b, its change in the velocity perpendicular to the direction of motion is

$$|\delta v_{\perp}| = \frac{2GM_{\mathrm{p}}}{b\Delta v} \,. \tag{1}$$

We then take advantage of the conservative nature of gravitational encounters to relate this to the parallel velocity by conservation of kinetic energy thus

$$\Delta v^2 = |\delta v_{\perp}|^2 + (\Delta v - \delta v_{||})^2 \,. \tag{2}$$

In the limit of small deflection angles (i.e., $\delta v_{||} \ll \Delta v$), this gives

$$\Delta v^2 \simeq \left(\frac{2GM_{\mathrm{p}}}{b\Delta v}\right)^2 + \Delta v^2 \left(1 - \frac{2\delta v_{||}}{\Delta v}\right) \,, \tag{3}$$

which reduces to

$$\delta v_{||} \simeq \frac{2G^2M_{\mathrm{p}}^2}{b^2\Delta v^3} \,. \tag{4}$$

If the orbital radius (semi-major axis) of the planet is a, then the change in specific angular momentum of the fluid element is

$$\Delta j = a.\delta v_{||} = \frac{2G^2M_{\mathrm{p}}^2 a}{b^2\Delta v^3} \,. \tag{5}$$

In a Keplerian system, the direction of the angular momentum exchange can be readily understood. Gas exterior to the planet's orbit feel a positive torque from the planet (as the planet has a higher orbital speed), so this exchange of angular momentum pushes the gas outwards and the planet inwards. Gas interior to the planet feels the opposite effect: it is pushes inwards by the planetary torque, and causes the planet to migrate outwards. The net direction of migration thus depends on the difference between the interior and exterior torques.

The total torque on the planet can be estimated by integrating Equation 5 over the entire disc. If we consider an annulus of gas exterior to the planet with surface density Σ and width db, the mass in the annulus is $dm = 2\pi\Sigma a db$. If the gas in this annulus has orbital frequency Ω and the planet has Ω_{p}, the timescale over which all of the gas in the annulus will encounter the planet is

$$\Delta t = \frac{2\pi}{|\Omega - \Omega_{\mathrm{p}}|} \,. \tag{6}$$

For small displacements $b \ll a$ we can approximate $|\Omega - \Omega_\mathrm{p}|$ as

$$|\Omega - \Omega_\mathrm{p}| \simeq \left|\frac{d\Omega_\mathrm{p}}{da}\right| b = \frac{3\Omega_\mathrm{p}}{2a} b\,. \tag{7}$$

We can therefore calculate the total torque on the planet as

$$\frac{dJ}{dt} = -\int \frac{\Delta j . dm}{\Delta t}\,. \tag{8}$$

We eliminate the Δv term by assuming near-Keplerian orbits, so that $\Delta v \simeq |\Omega_\mathrm{p}'|ab = (3/2)\Omega_\mathrm{p} b$. Substituting (and cancelling a lot of terms), we find that

$$\frac{dJ}{dt} = -\int_0^\infty \frac{8G^2 M_\mathrm{p}^2 \Sigma a}{9\Omega_\mathrm{p}^2 b^4} db\,. \tag{9}$$

This integral diverges, but if we specify some minimum impact parameter $b_\mathrm{min} > 0$ we find

$$\frac{dJ}{dt} = -\frac{8G^2 M_\mathrm{p}^2 \Sigma a}{27\Omega_\mathrm{p}^2 b_\mathrm{min}^3}\,. \tag{10}$$

In practice, values of b_min between the Hill radius (for low-mass planets) and the disc scale-height (for massive planets) give a torque which agrees approximately with that computed from more detailed analyses.

This simplified analysis captures many important features of the planet-disc interaction. We see that the the strength of the torque scales with the surface density Σ, so a more massive disc causes more rapid planet migration. We also see that dJ/dt scales with the square of the planet's mass. The planet's angular momentum scales linearly with M_p, so we conclude that more massive planets migrate more rapidly *if the disc conditions are the same*. Note that the second part of this sentence is crucial. As we will see shortly, massive planets can modify the local disc structure significantly (affecting both Σ and b_{min}), so it is not generally true that more massive planets migrate more rapidly.

1.1 Resonant Torques

The impulse approximation yields approximately the right answer in this case, but it is clear that this analysis has skated over the more subtle details of the problem. A full analysis instead considers the evolution of linear perturbations in a fluid disc, and was first applied to planet-disc interactions by Goldreich & Tremaine (1979, 1980). The first step is to decompose the perturbation to the Keplerian potential due to the planet into Fourier modes ($\propto \exp[im(\phi - \Omega_\mathrm{p} t)]$). The next step is to compute the response of the disc to these perturbations, and from this it is possible to calculate the torque on the planet. This procedure is decidedly non-trivial (as can be seen from the original papers), and subsequent work has refined the first results significantly.

The key result, however, is easy to state: the total angular momentum exchange between disc and planet can be expressed as the sum of the torques exerted at discrete resonances in the disc. These resonances correspond to the points in the disc where the planet excites waves, which almost invariably take the form of spiral density waves. Despite being not obvious from a mathematical perspective, this result intuitively makes sense: the torques at non-resonant locations in the disc do not "interfere" constructively, and consequently cancel out when averaged over many orbits[1].

[1] The non-trivial part of the problem essentially comes down to showing that the non-resonant torques cancel exactly, rather than just approximately.

In general, resonances occur when a characteristic frequency of the planet matches a frequency in the disc. If we consider only circular orbits there are two important types of resonance: **co-rotation** and **Lindblad** resonances. If the planet frequency is $\Omega_{\rm p}$ and the disc has orbital frequency $\Omega(R)$, the co-rotation resonance is found where

$$\Omega(R) = \Omega_{\rm p} \, . \tag{11}$$

If the disc is Keplerian (i.e., if we neglect gas pressure and self-gravity), then the co-rotation resonance is found at the planet's orbital radius. The condition for Lindblad resonances is similar:

$$m[\Omega(R) - \Omega_{\rm p}] = \pm\kappa(R) \qquad m \in \mathbb{Z} \, . \tag{12}$$

$\kappa(R)$ is formally the epicyclic frequency, but in a Keplerian disc $\kappa(R) = \Omega(R)$ and the Lindblad resonances are found at radii

$$R_{\rm L} = \left(1 \pm \frac{1}{m}\right)^{2/3} a \, . \tag{13}$$

A circular Keplerian disc therefore has a single co-rotation resonance, and a "comb" of Lindblad resonances that "pile up" close to the planet[2].

1.2 Type I migration

Type I planet migration refers to the migration of planets which are not massive enough to perturb the disc structure significantly. With this assumption in place, computing the total torque requires us only to sum up the contributions from the various different resonances

$$\Gamma = \sum \Gamma_{\rm ILR} + \sum \Gamma_{\rm OLR} + \sum \Gamma_{\rm CR} \, . \tag{14}$$

where the three terms respectively represent the sums of the partial torques from the inner and outer Lindblad resonances and the co-rotation resonance. In general the ILRs drive outward migration while the OLRs drive inward migration, and the result is largely independent of the surface density gradient. In practice, however, the first two terms are of comparable magnitude and opposing sign, so detailed calculations are required to determine Γ; simple analytic calculations struggle even to predict the direction of migration. Numerical calculations of Type I migration typically find rapid inward migration in laminar discs (on timescales $\ll 10^5$yr), but it is possible to choose a disc model in which the migration is outwards. Moreover, real discs are almost certainly turbulent, and the fluctuating torques that result from turbulent density fluctuations can dominate (e.g., Nelson & Papaloizou 2004). Consequently, although early work on Type I migration generally found very fast inward migration of terrestrial planets and giant planet cores, more recent calculations find a range of migration rates, with the outcome depending sensitively on both the density and temperature structure of the disc. Depending on the local disc properties, Type I migration may stall or even reverse direction, and much current research focuses on the role of so-called "planet traps" (locations in the disc where the migration rate is low). Determining the outcome of Type I migration remains a significant challenge: accurate calculations require high numerical precision and detailed knowledge of the protoplanetary disc, and outcomes can differ substantially between planetary systems.

1.3 Type II migration

We have already seen that the torque exerted on the disc by the planet scales $\propto M_{\rm p}^2$. As a result massive planets exert strong torques on the gas disc, and modify its structure significantly. Gas

[2] In the case of a planet on an eccentric orbit the number of resonances is greatly increased, though most do not contribute significantly to the torque.

interior to the planet's orbit loses angular momentum to the planet, while the exterior disc gas gains angular momentum. The combined effect of these torques is that gas is very efficiently expelled from the region near the planet, and a gap opens in the disc at the planet's radius. In a completely inviscid disc a gap can be opened by a planet with very low mass, but in a real disc angular momentum transport (viscosity) acts to diffuse material back towards the planet's radius. Whether or not a planet can open a gap therefore depends on whether the torques from the planet overcome the viscous torques in the disc.

The first requirement for gap opening is that the planet's Hill radius (its gravitational region of influence) must be at least comparable to the disc thickness[3] H. Expressing the Hill radius in terms of the planet-to-star mass ratio $q = M_{\rm p}/M_*$, we therefore require

$$Rq^{1/3} \gtrsim H \, , \tag{15}$$

and therefore

$$q \gtrsim \left(\frac{H}{R}\right)^3 \, . \tag{16}$$

For parameters typical of protoplanetary disc this implies that $q \gtrsim 10^{-4}$–10^{-3} is necessary for gap opening.

The second condition for gap opening is that the torque from the planet must exceed the viscous stress in the disc. In general this calculation requires detailed hydrodynamics, but several approximate arguments yield very similar answers. The calculations of Takeuchi et al. (1996) find that a gap will open in an α-disc if

$$q \gtrsim \left(\frac{c_{\rm s}}{a_{\rm p}\Omega_{\rm p}}\right)^2 \alpha^{1/2} \, . \tag{17}$$

If we make typical assumptions about the disc structure and assume $\alpha \simeq 0.01$, this again yields a critical planet mass $q \gtrsim 10^{-4}$–10^{-3}. We therefore see that for a solar-mass star, planets of Jupiter mass are expected to migrate in the Type II regime, while Neptune-mass objects undergo Type I migration. Saturn-mass planets lie close to the critical mass for gap-opening, and in practice are likely to open a partial gap in the disc. Computing the migration of planets in this regime is very complex, and the results depend crucially on how energy is dissipated in the so-called "horseshoe" region close to the planet.

Computing the migration rate in the Type II regime, however, is relatively straightforward. Once a gap has opened there is effectively no gas present at the resonances close to the planet, and consequently the inner and outer Lindblad torques are weak (and approximately cancel out). Unless the planet is extremely massive the disc dominates the angular momentum budget, so the planet (and its gap) simply migrate inwards with the disc's viscous accretion flow. This gives typical migration timescales of $\sim 10^5$yr at AU radii, and plausibly explains the existence of the so-called hot Jupiters (which are assumed to have formed at $\gtrsim$ 5–10AU and migrated inwards). Type II migration slows substantially for very massive planets (a few Jupiter masses or larger), as the planet's angular momentum is not negligible, but as long as the disc accretes the direction is invariably inwards. In single planet systems Type II migration is thought to be halted only by dispersal of the protoplanetary disc, and both analytic and "population synthesis" models[4]. find that Type II migration followed by "stranding" when the gas disc is dispersed successfully reproduces the observed distribution of $\sim$Jupiter-mass planets from $\sim$0.1–5AU (e.g., Armitage 2007; Alexander & Pascucci 2012).

[3]If this is not true the disc will continue to accrete across the planet's orbit at distances more than a Hill radius above or below the disc midplane.

[4]Population synthesis models employ simplified models or prescriptions in order to compute large ensembles of planet formation models and generate statistical distributions of predicted planet properties (see, e.g., the *Protostars & Planets VI* chapter by Benz. et al.)

2 Migration in planetesimal discs

We know from observations, both of the Solar System and of extra-solar debris discs, that a significant disc of solid bodies is left behind at the end of the gas disc's lifetime. These bodies (which we will refer to as planetesimals, even though some of them may be much more massive) continue to interact gravitationally with the planets, and can drive dynamical evolution of the system for at least hundreds of Myr.

The simplest case to consider here is that of a single planet of mass M_{p}, which orbits (at radius a) interior to a disc of planetesimals[5]. The planetesimal disc has surface density Σ_{p}, and no planetesimals are initially at $r < a$. Individual planetesimals are scattered inwards (i.e., on to lower angular momentum orbits) by gravitational encounters with the planet, and the planet migrates outwards (slightly) to conserve angular momentum[6]. We make the additional simplifying assumption that planetesimals do not interact with the planet again after they are scattered inwards.

Once again, migration is driven only by planetesimals with orbits which cross the Hill sphere of the planet. The total mass of planetesimals within this region is

$$\Delta m = 2\pi a \Sigma_{\mathrm{p}} r_{\mathrm{H}} \simeq 2\pi a^2 \Sigma_{\mathrm{p}} \left(\frac{M_{\mathrm{p}}}{M_*}\right)^{1/3} . \tag{18}$$

We estimate the change in specific angular momentum of a scattered planetesimal as the difference in angular momenta of circular orbits across this zone, so the total angular momentum change from scattering all of the planetesimals out of the zone is

$$\Delta J \simeq \Delta m \frac{1}{2}\sqrt{\frac{GM_*}{a}} r_{\mathrm{H}} . \tag{19}$$

Here the second term is the radial gradient of the specific angular momentum in circular orbits $[= d/da(\sqrt{GM_* a})]$. We assume that the planet remains on a circular orbit as it migrates, so

$$\Delta J \simeq M_{\mathrm{p}} \frac{dj}{da} . \Delta a = \frac{1}{2} M_{\mathrm{p}} \sqrt{\frac{GM_*}{a}} \Delta a . \tag{20}$$

Substituting for ΔJ, we find that the planet migrates by a distance

$$\Delta a \simeq \frac{2\pi a \Sigma_{\mathrm{p}} r_{\mathrm{H}}^2}{M_{\mathrm{p}}} . \tag{21}$$

In order to continue migrating, the planet must move far enough to encounter a new annulus of planetesimals before it exhausts the one at its original location. Fast migration therefore requires that $\Delta a \gtrsim r_{\mathrm{H}}$, which in turn requires that

$$M_{\mathrm{p}} \lesssim 2\pi a \Sigma_{\mathrm{p}} r_{\mathrm{H}} . \tag{22}$$

We see that rapid planetesimal-driven migration occurs if the planet mass is smaller than the mass of planetesimals within the gravitational region of influence. More massive planets may migrate slowly, but are unlikely to be significantly perturbed by planetesimals (though they will strongly influence the dynamics of the planetesimals).

[5]This configuration is relatively well-motivated, both by observations of the Solar System (which has a large number of trans-Neptunian objects) and by our understanding of planet formation (which suggests that the agglomeration of planetesimals into planets takes longer than the disc lifetime at large radii)

[6]Note that these large-angle scatterings have the opposite effect to the small-angle scattering we considered in the case of gas-driven migration: an exterior planetesimal disc causes outward migration of the planet.

Finally, we can estimate the rate of planetesimal-driven migration by noting that the time required to scatter all of the planetesimals out of the annulus is set (approximately) by the shear timescale across the annulus. Thus

$$\Delta t \simeq \left|\frac{d\Omega}{dR}\right|^{-1} \frac{1}{r_{\rm H}} \sim \frac{2}{3}\frac{a}{\Omega r_{\rm H}} \sim \frac{a}{r_{\rm H}} P \,, \tag{23}$$

where P is the local orbital period. Neglecting factors of order unity, the migration rate is therefore

$$\frac{da}{dt} \simeq \frac{\Delta a}{\Delta t} \simeq \frac{\Sigma_{\rm p} r_{\rm H}^3}{M_{\rm p} P} \simeq \frac{a}{P}\frac{\Sigma_{\rm p} a^2}{M_*} \,. \tag{24}$$

The second term here is the disc/star mass ratio, and the migration rate scales linearly with the mass of the planetesimal disc. Note also that as long as we are in the fast migration regime, the migration rate is independent of the planet's mass.

If we assume that a massive planetesimal disc exists at early times, this process can drive substantial migration of fairly massive planets. Modelling and observations both suggest that the total mass of such a disc can be 2–3 orders of magnitude larger than the mass of the present-day Kuiper Belt, and that planets as massive as Saturn can undergo significant migration. This picture is supported by Solar System observations, which show that a large number of Kuiper Belt objects are in 3:2 mean-motion resonance with Neptune. This sort of resonant trapping requires that Neptune migrated outwards significantly, sweeping up planetesimals into resonance as it did so, and models of this type have received significant interest in recent years. Of particular interest is the "Nice model", which shows how planetesimal-driven migration and resonance crossing can explain a number of different observed features in the Solar System. The same group later extended the Nice model to consider the so-called "Grand Tack" scenario. In this model Jupiter undergoes Type II migration and moves inwards to $\sim$1.5AU. At this point Saturn (which migrates more rapidly) becomes trapped in the 2:1 mean-motion resonance with Jupiter, and the two planets then migrate outwards again while remaining trapped in resonance. Jupiter's inward excursion depletes the disc of planetesimals and planetary embryos. The Grand Tack therefore plausibly accounts for the low mass of Mars (relative to the other terrestrial planets). but other aspects of the model remain highly idealised (most notably, gas accretion on to Saturn must be artificially suppressed in order to preserve its low mass relative to Jupiter). Nevertheless, this picture reflects the general trend towards increasingly dynamic models of planetary system evolution.

Further Reading

In addition to the main list of references given on the course home-page, the following papers are particularly relevant to this lecture:

Kley & Nelson, *Planet-Disk Interaction and Orbital Evolution*, 2012, ARA&A, 50, 211.
Baruteau et al., *Planet-Disc Interactions and Early Evolution of Planetary Systems*, Protostars & Planets VI, 2014, p667 (`arXiv:1312.4293`).
Benz et al., *Planet Population Synthesis*, Protostars & Planets VI, 2014, p691 (`arXiv:1402.7086`).
Davies et al., *The Long-Term Dynamical Evolution of Planetary Systems*, Protostars & Planets VI, 2014, p787 (`arXiv:1311.6816`).
Raymond et al., *Terrestrial Planet Formation At Home And Abroad*, Protostars & Planets VI, 2014, p595 (`arXiv:1312.1689`).
Papaloizou et al., *Disk-Planet Interactions During Planet Formation*, 2007, Protostars & Planets V, p655.
Levison et al., *Planet Migration in Planetesimal Disks*, 2007, Protostars & Planets V, p669.
Morbidelli, *Dynamical Evolution of Planetary Systems*, 2013, chapter in "Planets, Stars and Stellar Systems" (`arXiv:1106.4114`).
Goldreich & Tremaine, *The excitation of density waves at the Lindblad and corotation resonances by an external potential*, 1979, ApJ, 233, 857.
Goldreich & Tremaine, *Disk-satellite interactions*, 1980, ApJ, 241, 425.
Lin & Papaloizou, *Tidal torques on accretion discs in binary systems with extreme mass ratios*, 1979, MNRAS, 186, 799.
Nelson & Papaloizou, *The interaction of giant planets with a disc with MHD turbulence - IV. Migration rates of embedded protoplanets*, 2004, MNRAS, 350, 849.
Takeuchi et al., *Gap Formation in Protoplanetary Disks*, 1996, ApJ, 460, 832.
Walsh et al., *A low mass for Mars from Jupiter's early gas-driven migration*, 2011, Nature, 475, 206. (This is the so-called "Grand Tack" model.)

Finally, the following three papers are collectively referred to as the "Nice model" of the Solar System, named after the French city where much of the work was done:

- Tsiganis et al., *Origin of the orbital architecture of the giant planets of the Solar System*, 2005, Nature, 435, 459
- Morbidelli et al., *Chaotic capture of Jupiter's Trojan asteroids in the early Solar System*, 2005, Nature, 435, 462
- Gomes et al., *Origin of the cataclysmic Late Heavy Bombardment period of the terrestrial planets*, 2005, Nature, 435, 466