Lecture 4: Planet formation

1 Formation of planetary cores / terrestrial planets

Following on from the Lecture 3, our (somewhat dubious) starting point is a protoplanetary disc populated by a large swarm of planetesimals. Planetesimals are (by definition) sufficiently massive that collisions between pairs of them have sufficient gravity that most of the mass of the colliding bodies ends up agglomerating into a single body[1]. Terrestrial (proto-)planets and giant planet cores are the end products of this process of collisional growth, and the dominant factors in controlling the growth rate are the collision cross-section and the size of the available mass reservoir.

For small bodies, as we saw in Lecture 3, the relevant cross-section is simply the (projected) area of the particle. For planetesimals, however, we must consider the additional effects of gravitational focusing. We consider two bodies of radius s and mass m, approaching each other on parallel trajectories with relative velocity σ and impact parameter b. As they approach they will be deflected towards one another by their mutual gravity, and we can equate energy in the initial state with that at closest approach thus

$$2\,.\,\frac{1}{2}\,(m)\left(\frac{\sigma}{2}\right)^2 = 2\,.\,\frac{1}{2}mv_{\rm max}^2 - \frac{Gm^2}{\Delta R}\,. \tag{1}$$

$v_{\rm max}$ is the velocity of each body at closest approach, where the planetesimals have a minimum separation ΔR. Note also that we have neglected the potential energy in the initial (widely separated) configuration. Angular momentum conservation during the interaction gives

$$\frac{1}{2}v_{\rm max}\Delta R = \frac{1}{2}b\,.\,\frac{1}{2}\sigma \tag{2}$$

$$v_{\rm max} = \frac{b}{2\Delta R}\sigma\,. \tag{3}$$

We require $\Delta R < s$ for the bodies to collide, so if we substitute this expression for $v_{\rm max}$ in Equation 1 and set $\Delta R = s$, we find that the largest impact parameter that leads to a collision is

$$b^2 = s^2 + \frac{4Gms}{\sigma^2}\,. \tag{4}$$

This is typically written in terms of the escape velocity from the point of contact $v_{\rm esc}^2 = 4Gm/s$, so we can write the cross-section as

$$\pi b^2 = \pi s^2\left(1 + \frac{v_{\rm esc}^2}{\sigma^2}\right) = \pi s^2(1+\Theta)\,. \tag{5}$$

The gravitational focusing factor $\Theta = v_{\rm esc}^2/\sigma^2$ is often referred to as the Safronov number. The collision cross section therefore increases dramatically if the planetesimal disc is dynamically "cold" (i.e., the velocity dispersion σ is low).

We now compute the growth rate due to collisions in a disc of planetesimals with velocity dispersion σ and surface density $\Sigma_{\rm p}$. We assume that the velocity dispersion is isotropic, so the

[1]For the purposes of this discussion we will assume that planetesimal collisions result in agglomeration, but in doing so we have glossed over many details. In particular, the efficiency of mass retention depends on the material strength of the colliding bodies, and many collisions have sufficient energy to fully disrupt the bodies. This is a topic of continuing research, and recent calculations (e.g., Stewart & Leinhardt 2009) suggest that gravity "wins" only for planetesimals of size $\gtrsim$ 10km.

planetesimal disc has half-thickness $H_{\rm p} \simeq \sigma/\Omega$ and the volume density of planetesimals is $\rho_{\rm p} = \Sigma_{\rm p}/H_{\rm p} = \Sigma_{\rm p}\Omega/\sigma$. The growth rate due to collisions is therefore

$$\frac{dm}{dt} = \rho_{\rm p}\sigma\pi b^2 \simeq \Sigma_{\rm p}\Omega\pi s^2(1+\Theta)\,. \quad (6)$$

For a sufficiently low velocity dispersion we can assume that $\Theta \gg 1$ (i.e., we assume that gravitational focusing dominates the cross-section), so

$$\frac{dm}{dt} \simeq \Sigma_{\rm p}\Omega\pi s^2\Theta = \pi\Sigma_{\rm p}\Omega s\frac{4Gm}{\sigma^2}\,. \quad (7)$$

The particle mass $m \propto s^3$, so we can substitute to find

$$\frac{dm}{dt} \propto m^{4/3}\,. \quad (8)$$

If we integrate this expression we find that $m(t)$ diverges to infinity in a finite time. This phenomenon is known as runaway growth, and demonstrates that if gravitational focusing dominates then the growth of planetesimals can be very rapid.

In practice a few different factors act to limit the growth rate. The first is the planetesimal velocity dispersion σ. In a real system the number of close encounters exceeds the number of physical collisions, and repeated close encounters tend to increase the velocity dispersion and limit the efficiency of gravitational focusing. Gas drag can damp the velocity dispersion in some cases, but the limit of $\Theta \gg 1$ is not always reached. In addition, the collision velocities can be large enough to cause catastrophic disruption of the colliding bodies, and in these circumstances collisional growth may be rather inefficient.

Even if runaway growth does occur, however, it can only proceed until the local mass reservoir has been exhausted. Planetesimals can only accrete from within their gravitational region of influence (the Hill radius), given by $r_{\rm H} = a(m/M_*)^{1/3}$ (where a is the orbital radius). The total mass of planetesimals in this "feeding zone" is therefore

$$2\pi a r_{\rm H}\Sigma_{\rm p} = 2\pi a^2\left(\frac{m}{M_*}\right)^{1/3}\Sigma_{\rm p}\,. \quad (9)$$

The maximum attainable mass is achieved by accreting all of the planetesimals in the feeding zone, and this so-called isolation mass is therefore given by

$$m_{\rm iso} \simeq 2\pi a^2\left(\frac{m_{\rm iso}}{M_*}\right)^{1/3}\Sigma_{\rm p}\,. \quad (10)$$

Neglecting factors of order unity, we find that

$$m_{\rm iso} \simeq \Sigma_{\rm p}^{3/2}a^3M_*^{-1/2}\,. \quad (11)$$

For realistic disc models $m_{\rm iso}$ is therefore an increasing function of the orbital radius a, so we expect more massive solid bodies to form at larger orbital radii. If we substitute typical parameters we find that the isolation mass is small in the terrestrial planet zone ($\sim 0.1{\rm M}_\oplus$), which suggests that isolation plays an important role in the formation of terrestrial planets. Indeed, the final assembly of Earth-like planets probably requires the agglomeration of several isolation-mass objects into a single planet. Models suggest that terrestrial planet formation via repeated giant impacts of isolation-mass objects takes ~ 100Myr, which is broadly consistent with Solar System observations.

By contrast, estimates of the isolation mass at $\gtrsim 5$AU are typically $\gtrsim 10{\rm M}_\oplus$, comparable to (or greater than) the core masses of the Solar System giant planets. This suggests that isolation may never actually occur in the formation of giant planets, and that their cores can plausibly be built through collisional growth alone.

2 Core accretion

Once we have formed solid planetary cores, the question which then arises is how (and if) the core can subsequently accrete gas from the disc. The crudest statement we can make is that in order to retain any gaseous envelope, the escape speed at the core surface must exceed the sound speed in the gas. The escape speed from the surface of a single spherical body is

$$v_{\rm esc} = \sqrt{\frac{2Gm}{s}}\,, \tag{12}$$

and $m = (4\pi/3)\rho_{\rm d}s^3$ where, as before, $\rho_{\rm d}$ is the material density of the solid core. If we write the sound speed in terms of the disc thickness

$$c_{\rm s} = \frac{H}{R}v_{\rm K}\,, \tag{13}$$

then setting $c_{\rm s} = v_{\rm esc}$ tells us that bodies will be able to accrete gas if

$$m \gtrsim \left(\frac{3}{32\pi}\right)^{1/2}\left(\frac{H}{R}\right)^3 \frac{M_*^{3/2}}{\rho_{\rm d}^{1/2}a^{3/2}}\,. \tag{14}$$

If we assume canonical numbers for a disc at 5AU ($H/R = 0.1$, $\rho_{\rm d} = 1\,{\rm g\,cm^{-3}}$), then the limiting mass is $m \gtrsim 4 \times 10^{-3}{\rm M}_\oplus$. However, at this limit the planet will retain only a very tenuous gas envelope; more detailed calculations require consideration of the hydrostatic structure of the accreting envelope. This depends sensitively on a number of parameters (notably the opacity of the accreting gas), and requires detailed calculations. The first self-consistent models of this process were computed by Pollack et al. (1996), and many subsequent calculations have verified their basic result. This is the so-called "core accretion" model for giant planet formation, which has three distinct phases:

Core formation
This is the process discussed in Section 1. A solid core undergoes runaway growth, and the mass of the planet is dominated by the core until it approaches its isolation mass. This phase is relatively short ($\sim 10^5$yr), and core formation therefore occurs on a time-scale much shorter than the ∼1–10Myr gas disc lifetime.

Hydrostatic growth
In this phase the planet accretes gas from the disc, and the envelope grows in hydrostatic equilibrium. However, continued accretion of gas requires the planet to contract, primarily as a result of radiative cooling. While the solid core dominates the mass the planet can only contract gradually (as only the envelope is compressible), and consequently accretion during this phase is slow. This gradual increase in the planet mass increases the size of the feeding zone, allowing continued accretion of planetesimals (and therefore the core mass also grows). This phase typically lasts ∼Myr, and ends when the envelope mass approaches the core mass.

Runaway accretion
Once the gaseous envelope starts to dominate the planet's mass the rate of accretion increases dramatically, and we see runaway growth of the envelope. The accretion is initially limited only by the rate at which the disc can feed gas to the planet, and growth is very rapid. This phase is short, but increases the planet mass by an order of magnitude or more. Runaway accretion is terminated by the dispersal of the gas disc, or for massive planets accretion can be shut off by local tidal effects (torques from the planet on the disc). Once accretion ceases the "proto-planet" undergoes gradual Kelvin-Helmholz contraction to reach its final, equilibrium structure.

Core accretion is a viable mechanism for the formation of gas giant planets, and has been applied very successfully to both Solar System formation and the formation of exo-planets. Various "flavours" of the core accretion scenario exist, making somewhat different assumptions, but all follow the same qualitative picture. The major technical uncertainty in these models is the magnitude of the opacity of the accreting gas. The thermal structure of the envelope depends critically on the opacity, but because most of the opacity is due to dust the precise value of κ is difficult to estimate. The resulting uncertainty in the accretion rate translates into a factor-of-several uncertainty in the growth time-scale, with lower opacities promoting faster accretion. In addition, Type I migration (see Lecture 5) is expected to be very important for objects in this size range, and it may be that cores migrate out of the disc before they are able to accrete substantial envelopes.

Beyond these uncertainties, there are two major concerns about the viability of the core accretion model. The first is that it assumes a starting point (a disc populated with copious planetesimals) that may not be realistic. We know from Solar System studies (and debris disc observations) that planetesimal-size objects are common, but as we saw in the last lecture their formation remains something of a mystery. The second sticking point is the long duration of the hydrostatic phase. The initial gas accretion rate is low, and for typical assumptions (about disc structure and opacity) it takes 1–10Myr for planets to become massive enough to undergo runaway gas accretion. This time-scale is uncomfortably close to the typical lifetimes of protoplanetary discs, though recent developments suggest that pebble accretion (see Lecture 3) may speed up growth significantly during this phase. This still requires that planetesimals and solid cores can form rapidly at the beginning of the disc lifetime, however, and even with efficient pebble accretion the length of the hydrostatic growth phase can remain problematic. Nevertheless, core accretion remains the most plausible model for the formation of most of the planets that we observe.

3 Gravitational instability

An alternative to the process of core accretion is the idea that planets can form directly from gravitational fragmentation of the disc. This idea has a long history, but was re-invented in its modern form by Boss (1997). Formation of planets in this manner sidesteps many of the problems associated with the "standard" core accretion theory we have described above, but instead raises a number of other issues.

We first consider the basic physics of gravitational instability in gaseous discs. A formal approach considers the response of the disc to a small gravitational perturbation (see, e.g., Chapter 12 of Pringle & King), but a much simpler analysis yields the same qualitative result. If we consider a patch of a disc of size l, then the mass of the patch is $\Delta M \simeq \Sigma l^2$ and the gravitational potential energy of the patch (due to its own gravity) is

$$U_G \simeq -\frac{G\Delta M^2}{l} \simeq -G\Sigma^2 l^3 \, . \tag{15}$$

The thermal energy is

$$U_T \simeq \frac{1}{2}\Delta M c_s^2 \simeq \frac{1}{2}\Sigma l^2 c_s^2 \, , \tag{16}$$

and the rotational kinetic energy is

$$U_R \simeq \frac{1}{2}\Delta M \Omega^2 l^2 \simeq \frac{1}{2}\Sigma\Omega^2 l^4 \, . \tag{17}$$

If the disc is to become unstable we require that gravity overcome pressure and rotational support, so we require that

$$U_G + U_R + U_T < 0 \, . \tag{18}$$

Substituting, we find that

$$-2G\Sigma^2 l^3 + \Sigma\Omega^2 l^4 + \Sigma l^2 c_s^2 = \Sigma l^2(-2G\Sigma l + \Omega^2 l^2 + c_s^2) < 0 \, . \tag{19}$$

The term in brackets is quadratic in l, and has a minimum at $l = G\Sigma/\Omega^2$. The condition for instability is met if the left-hand side is negative at this minimum, and therefore if

$$\frac{c_s\Omega}{G\Sigma} < 1\,. \tag{20}$$

This is approximately the famous Toomre (1964) condition for gravitational instability in a disc. A more rigorous analysis yields the result that the disc is unstable to axisymmetric perturbations if

$$Q = \frac{c_s\Omega}{\pi G\Sigma} < 1\,. \tag{21}$$

The term on the left is usually referred to as the "Toomre Q" parameter, and lower values of Q give rise to instability. This form intuitively makes sense: increasing the temperature or the rotation rate increases Q and stabilises the disc, while increasing the disc mass lowers Q and makes the disc more unstable.

We can use the Toomre criterion to estimate whether or not protostellar discs are likely to be gravitationally unstable. The disc thickness $H \simeq c_s/\Omega$, the orbital frequency $\Omega^2 \simeq GM_*/R^3$ [2], and the disc mass $M_d \simeq \pi\Sigma R^2$. We can therefore re-arrange the Toomre criterion to find that the disc will be unstable if

$$\frac{H}{R} \lesssim \frac{M_d}{M_*}\,. \tag{22}$$

As we have seen protoplanetary discs typically have $H/R \sim 0.1$, so we therefore require that the disc be at least $\sim 10\%$ of the stellar mass in order to be unstable[3]. This is close to the upper limit of observed disc masses, but it is not implausibly large. Indeed, as most of the stellar mass must have been accreted through the disc at some point, most discs probably are massive enough to become gravitationally unstable at large radius during the early stages of their evolution.

From this analysis we can also make a crude estimate of the typical fragment (planet) masses which result from gravitational instability. As we have previously noted, the mass of the unstable patch

$$\Delta M \simeq \Sigma l^2\,, \tag{23}$$

and the characteristic length-scale for the instability is $l = G\Sigma/\Omega^2 \sim H$. We can therefore substitute to find

$$\Delta M \simeq \Sigma H^2 = \left(\frac{H}{R}\right)^2 \Sigma R^2\,. \tag{24}$$

If we then take $M_d \sim \Sigma R^2$, we can substitute from Equation 22 [$M_d \sim (H/R)M_*$] to find

$$\Delta M \sim \left(\frac{H}{R}\right)^3 M_* \sim 1\mathrm{M_{Jup}}\,, \tag{25}$$

(assuming a $1\mathrm{M}_\odot$ star and $H/R \simeq 0.1$). Note, however, that this is merely the initial fragment mass, and bound objects continue to accrete mass from the disc. This highlights an important feature of the gravitational instability model: it favours very massive planets. Also, as the planets form directly from the protoplanetary disc they are likely to be gas rich. This may therefore provide an alternative mechanism for the formation of gas giant planets, but the gravitational instability model (in its simplest form) is not likely to form low-mass or terrestrial planets.

[2] Note that both of these equations are now only approximate, the disc's contribution to the gravitational potential is no longer negligible and the orbits are not strictly Keplerian. However, the near-Keplerian approximation is a good one as long as $M_\mathrm{d} \ll M_*$.

[3] This is only an approximate condition. In particular, we should note that the Toomre criterion is a local condition, while in Equation 22 we have expressed it in terms of global properties of the system (disc mass). In a real system Q is a function of both position and time, and varies significantly across the disc.

3.1 Thermodynamics and fragmentation

An important additional consideration is whether or not a gravitationally unstable disc will actually fragment. We first note that, formally, the Toomre criterion tells us when the disc will become unstable to axisymmetric perturbations. However, a shearing disc becomes unstable to non-axisymmetric perturbations at slightly larger values of Q ($Q \lesssim 1.5$–2), so we expect the initial development of the gravitational instability (GI) to be in the form of spiral density waves in the disc. These spiral density waves induce Reynolds and gravitational stresses in the disc, and these stresses can transport angular momentum and drive accretion[4].

Spiral density waves can transport angular momentum, but if we consider their behaviour in detail it becomes clear that the development of GIs depends critically on the disc thermodynamics. From the Toomre criterion we see immediately that colder discs are more unstable. However, the instability results in heating, initially through adiabatic contraction of the unstable gas and and later through the weak shocks induced by the spiral density waves. This in turn drives the disc back towards stability, so the disc can only remain unstable if it is able to cool efficiently. We therefore expect the disc to be able "self-regulate" towards a state where GI-induced heating balances cooling, and $Q \simeq 1$. Numerical simulations have shown that discs can attain such a self-regulated state, and that discs in this state can drive quasi-steady angular momentum transport over long timescales (at least hundreds of orbital periods).

If we assume that the angular momentum transport and heating from GIs both occur locally (i.e., that the GI behaves like an alpha-disc, with no wave-like transport of energy)[5], and that the disc self-regulates to thermal equilibrium, we can derive an expression for the disc's cooling time-scale. The thermal energy per unit area of the disc can be written as

$$U_{th} = \frac{1}{\gamma(\gamma-1)} \Sigma c_s^2 \, , \tag{26}$$

where γ is the ratio of specific heats of the gas. ($\gamma = 5/3$ for a monatomic gas, or $7/5$ for a diatomic gas.) In a Keplerian disc viscous heating (per unit area) occurs at a rate

$$\frac{dU}{dt} = \frac{9}{4}\nu\Sigma\Omega^2 = \frac{9}{4}\alpha_g c_s^2 \Sigma\Omega \, , \tag{27}$$

where α_g refers to the "effective alpha" induced by the GI (through Reynolds and gravitational stresses; see Lecture 2). If heating balances cooling, we therefore require the disc to cool on a time-scale

$$t_{\rm cool} = \frac{U_{th}}{dU/dt} = \frac{4}{9\gamma(\gamma-1)} \frac{1}{\alpha_g \Omega} \, . \tag{28}$$

Alternatively, one can parametrize the effective transport induced by GIs as

$$\alpha_g = \frac{4}{9\gamma(\gamma-1)} \frac{1}{t_{\rm cool}\Omega} \, . \tag{29}$$

In principle, this equation suggests that GIs can give rise to arbitrarily large values of α_g, and therefore transport angular momentum at very large rates. This is not the case, however, as we can see by once again considering our unstable patch of disc. If cooling balances compressional heating the patch is "quasi-stable", but if the cooling rate becomes large (i.e., $t_{\rm cool}$ becomes short) the gas must contract very rapidly in order to maintain thermal equilibrium. For sufficiently rapid cooling, pressure can no longer support the collapsing gas against (self-)gravity and the disc

[4] It is left as an exercise for the enthusiastic student to show that only trailing spiral waves transport angular momentum outwards.

[5] Numerical simulations show that this approximation holds for low disc masses, $M_d \lesssim 0.2M_*$; above this we see significant power in low-order spiral modes (such as $m = 2$). These modes are typically short lived, leading to outbursts of accretion, and massive discs are generally not able to attain a self-regulated state.

fragments. Numerical simulations find that the "fragmentation boundary" occurs at $t_{\rm cool}\Omega \sim 5$–10, which in turn suggests that the maximum efficiency of angular momentum transport by GIs (in the local limit) is $\alpha_g \sim 0.1$.

In order for discs to fragment, we therefore have two conditions: (i) the disc must be massive and/or cold enough to satisfy the Toomre criterion; and (ii) the local (radiative) cooling rate must be high. Whether or not these conditions are ever satisfied in real discs remains a matter of some debate, but some aspects of this problem have now become clear. The effects of stellar irradiation mean that $Q \gg 1$ at small radii, and for discs around solar-mass stars gravitational instability is invariably negligible in the inner $\sim$ 10AU, even in massive discs. At larger radii ($\gtrsim$ 50AU) the most massive protostellar discs probably are gravitationally unstable, but it is not clear whether or not they can cool rapidly enough to fragment. If the cooling time-scale is long then spiral waves can drive rapid angular momentum transport, and this process is probably responsible for most protostellar accretion at early times.

If instead the disc cools rapidly enough to fragment, the immediate outcome is the formation of rather massive fragments ($\gtrsim 1{\rm M_{Jup}}$) at large radii ($\gtrsim$ 50AU). The subsequent evolution of these fragments depends critically on the thermodynamics of both the disc and the collapsing "protoplanets". Most models find that cooling remains efficient after fragmentation, so any fragments that form contract rapidly and continue to accrete gas from the disc (akin to the runaway phase of core accretion). This leads to the formation of very massive objects, often in the brown dwarf regime ($\gtrsim 12{\rm M_{Jup}}$). The dynamical evolution of objects formed via disc fragmentation remains uncertain, with rapid migration, dynamical ejection, and coalescence all likely to play a role. Current numerical simulations disagree on the relative importance of these processes, and some other key physical effects (such as infall on to the disc, which is almost certainly important during this epoch) remain poorly understood. However, most current models suggest that where fragmentation does occur, gravitational instability is more likely to form brown dwarfs or stellar companions than objects of planetary mass (e.g., Kratter & Lodato 2016). Observational evidence here is limited, but recent ALMA data also points towards brown dwarf- or stellar-mass objects as the outcome of disc fragmentation (Tobin et al. 2016).

Further Reading

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